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# Explosion, Waves of Resurgence or Convergence? Macroeconomic and Epidemiological Dynamics of the COVID-19 Pandemic

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# Explosion, Waves of Resurgence or Convergence? Macroeconomic and Epidemiological Dynamics of the COVID-19 Pandemic

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## Abstract

Using a simple model, this paper attempts to theoretically explore the dynamics of interaction between the epidemiological processes of the COVID-19 pandemic and the macroeconomic processes which influence the non-pharmaceutical policy interventions to control the pandemic. We show that convergence to a steady state where the pandemic is under control requires specific form of policy interventions. A failure to achieve this might result in either explosive dynamics of uncontrolled pandemic and economic collapse or a limit cycle from Andronov-Hopf bifurcation, resulting in repeated waves of resurgence in pandemic and very short-run fluctuations in economic activities. Broad stylized empirics of the pandemic resembles the outcome from the simple model.

*Keywords:* COVID-19, health, lockdown, stability, Andronov-Hopf bifurcation, policy intervention .

*JEL classification:* E10, E61, I10, I18.

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# 1 Introduction

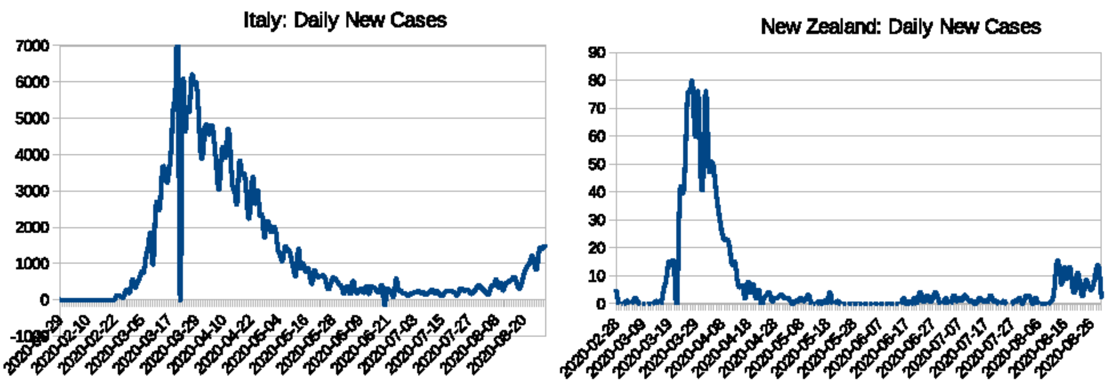
COVID-19 is a highly infectious disease, caused by the SARS-Cov-2 virus, which is often transmitted by asymptomatic individuals (World Health Organization 2020*a*). It was initially detected in the Wuhan province of China but quickly spread to the rest of the world. The World Health Organization declared this to be a global pandemic on March 11, 2020 (World Health Organization 2020*b*). The rapid progress of the pandemic since then has also led to the emergence of a large literature on formal mathematical models of the dynamics of COVID-19 pandemic (see, for instance, Thomas, Sturdivant, Dhurandhar, Debroy & Clark 2020, Anirudh 2020, Ndaïrou, Area, Nieto & Torres 2020, Ivorra, Ferrández, Vela-Pérez & Ramos 2020, Kucharski, Russell, Diamond, Liu, Edmunds, Funk, Eggo, Sun, Jit, Munday, Davies, Gimma, van Zandvoort, Gibbs, Hellewell, Jarvis, Clifford, Quilty, Bosse, Abbott, Klepac & Flasche 2020, Badr, Du, Marshall, Dong, Squire & Gardner 2020). Many of these models owe their origins to the mathematical epidemiological models, described, for instance, by Brauer, Castillo-Chavez & Feng (2019). These models divide the total population into categories such as susceptible to infection (S), exposed to infection (E), infected (I), quarantined (Q), recovered from infection (R) apart from those who die due to diseases such as COVID-19. This gives rise to classes of epidemiological models such as the SIR, SEIR, SIQR etc.

In the absence of a effective vaccine to control the pandemic, non-pharmaceutical policy interventions have emerged as important tools to control the spread of the pandemic. These non-pharmaceutical interventions, however, have implications beyond the ones which are usually included in the traditional epidemiological models. For instance, an important aspect of the non-pharmaceutical policy interventions to control the COVID-19 pandemic has been their impact on the economic activities. Many of these interventions, like travel restrictions or restrictions on economic activities (including lockdowns, restrictions on public transport and limits on public gatherings etc.) have adversely affected economic indicators like employment rates. To understand the progress of the pandemic and the effectiveness of the policy response to control it, we need to understand the dynamics of the interaction between the epidemiological and the macroeconomic processes. The policymakers, however, are expected to respond not only to the pandemic, but also to the adverse economic impact of a combination of the pandemic as well as the non-pharmaceutical policy interventions, which affect the livelihood of the citizens in the affected countries. This is an area which has been relatively less explored in the formal mathematical literature on COVID-19 pandemic.[1] The current paper presents a preliminary attempt to

[1]There has been some recent literature in macroeconomics which attempts to analyze the economic impact of the pandemic and restrictive policy responses in a neoclassical or New Keynesian framework; see, for instance Guerrieri, Lorenzoni, Straub & Werning (2020) and Eichenbaum, Rebelo & Trabandt (2020). This line of literature, however, does not explicitly incorporate the epidemiological dynamics of the pandemic. Instead it considers the pandemic as an exogeneous shock, and

address this concern.

The broad stylized facts on the progress of the pandemic reflects the outcome of this interaction between the epidemiological and economic processes. We find that the progress has varied from country to country. The policy interventions have managed to control or stabilize the spread of the pandemic in some countries whereas in some countries, it has generated waves or cycles of transmission. For instance, in Italy (which was one of the earliest countries to suffer a severe wave) and New Zealand, the non-pharmaceutical policy interventions have succeeded so far in stabilizing the number of daily new cases and the number of actively infected persons after an initial peak in outbreak (figure) 1). In other countries like Australia, Israel, Japan, Serbia, Spain and Vietnam, there have been multiple waves of outbreak (as represented by the number of daily new cases or the number of persons actively infected with COVID-19, see figure 2). In many of these countries, re-emergence of the outbreak has often been accompanied with gradual lifting of restrictions on economic activities which were earlier put into place to control the pandemic.




(a) Daily new cases in Italy

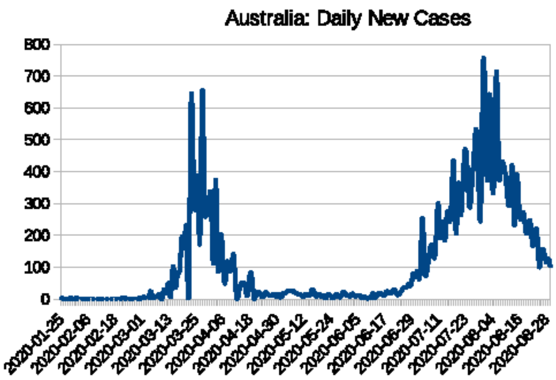
(b) Daily new cases in New Zealand

Figure 1: Effective control of pandemic

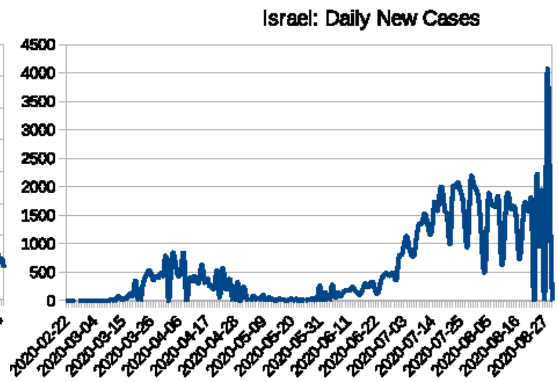
Source: World Health Organization COVID-19 Dashboard

In the following sections, we attempt to theoretically explore the dynamics of interaction between the epidemiological process of progress of the pandemic and the macroeconomic processes. We begin with a simple general model of interaction between the epidemiological and macroeconomic processes in section 2, followed by a more specific model. We discuss the results and comparative dynamics from the model in section 3, and conclude in section 4.

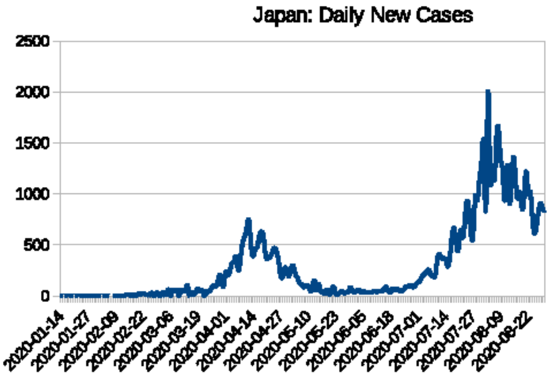
therefore, fails to incorporate the dynamics of the interaction between the epidemiological and the macroeconomic processes.



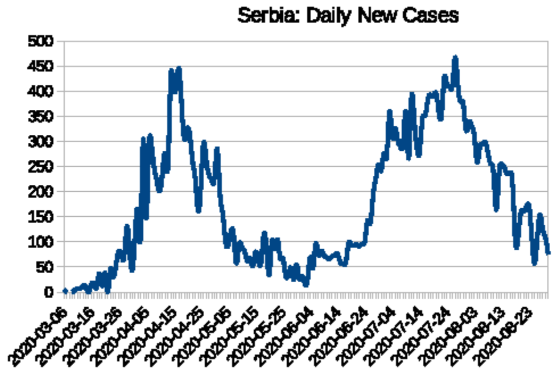

(a) Daily new cases in Australia



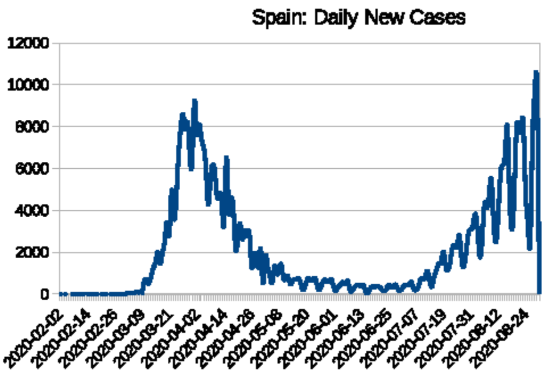

(b) Daily new cases in Israel



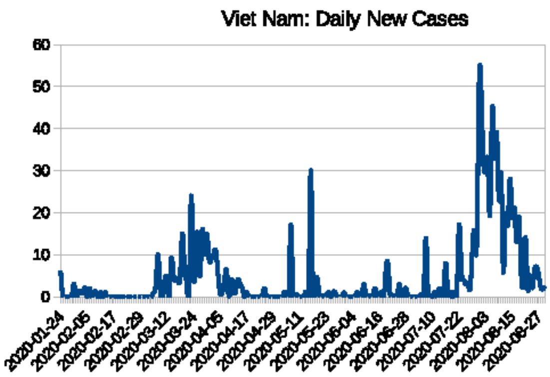

(c) Daily new cases in Japan




(d) Daily new cases in Serbia




(e) Daily new cases in Spain




(f) Daily new cases in Vietnam

Figure 2: Multiple waves

Source: World Health Organization COVID-19 Dashboard

# 2 The Model

## 2.1 Basic Setup

Consider a general model of a closed economy before it is affected by the pandemic. The aggregate output per unit of *effective labor* at time $t$, $y \in \Re_+$, grows according to the following rule:

$$\frac{\dot{y}}{y} = f(y), \quad f : \Re_+ \to \Re_+, \; f(0) > 0, \; f'(y) < 0 \tag{1}$$

where $f$ is continuously differentiable. Once the population is hit by COVID-19 pandemic, the normal economic activities are adversely affected, both due to the pandemic itself and due to policy-induced restrictions on economic activities to control the spread of infections. The extent of the adverse impact depends on the progress of the pandemic. We formalize this as follows. Let the proportion of population actively affected by COVID-19 be represented by $i \in \Re_+$. This leads to a modification of (1) as follows:

$$\frac{\dot{y}}{y} = \alpha f(y, i), \quad f : \Re_+ \times \Re_+ \to \Re_+, \; f(0,0) > 0, \; f_y < 0, \; f_i < 0 \tag{2}$$

where $f$ is *continuously differentiable* and $\alpha > 0$ is a speed of adjustment parameter. $f_i$ depends, among other things, on the policy environment. The more quickly the policymaker responds to the spread in the pandemic with restrictions on economic activities, higher will be the sensitivity of economic activities to the rate of infection (resulting in a larger negative value of $f_i$.)

Next, we turn our attention to the dynamics of spread of COVID-19. The dynamics of pandemics such as COVID-19 are described by a class of mathematical models described, for instance by Brauer et al. (2019). These models divide the total population into categories such as susceptible to infection (S), exposed to infection (E), infected (I), quarantined (Q), recovered from infection (R) apart from those who die due to diseases such as COVID-19. This gives rise to classes of epidemiological models such as the SIR, SEIR, SIQR etc. However, in the absence of reliable evidence on whether those infected with COVID-19 develop resistance, we here consider a simpler model of infection dynamics. At low rates of infection, the infected can easily be identified, quarantined and prevented from infecting others. Despite being highly infectious, COVID-19 so far has a relatively high recovery rate, unlike say, Ebola or SARS-2003. This means that as long as the proportion of population affected with COVID-19 remains low, the existing healthcare system in any country can ensure that the infected persons recover without infecting others. However, with an increase in the proportion of infected people, it is more likely for the infected persons to come in contact with others and spread the infection (see, for instance, Shen & Bar-Yam 2020).[2] This can lead to an exponential growth in the transmission of the disease.

[2]In the long-run, for many diseases this process might be reversed by development of herd immunity; however, this argument is not applicable when one is in the middle of the pandemic. In

The spread of infection is further helped by normal economic activities. Economic activities increase the possibilities of the infected persons coming into contact with uninfected persons, positively affecting the proportionate rate of growth of infection. This might be represented in the form of the following differential equation:

$$\frac{\dot{i}}{i} = \beta g\left(y, i\right), \quad g : \Re_{+} \times \Re_{+} \rightarrow \Re_{+},\ g\left(0,0\right) < 0,\ g_y > 0,\ g_i > 0 \tag{3}$$

where $g$ is *continuously differentiable* and $\beta > 0$ is a speed of adjustment parameter. $g_y$ is the sensitivity of the rate of infection to increase in economic activity.

Equations (2) and (3) might be considered together as a system of differential equations, representing the dynamic interaction between the economic activities and the spread of infection in the economy. We write it below for clarity:

$$\begin{aligned} \dot{y} &= \alpha f\left(y, i\right) y \\ \dot{i} &= \beta g\left(y, i\right) i \end{aligned} \tag{4}$$

The dynamical system represented by (4) belongs to the broad class of 'predator-prey' or 'Kolmogorov-Lotka-Volterra' class of models, originally formulated by Lotka (1925) and Volterra (1927) in a biochemical and ecological application respectively, and later on generalized by Kolmogorov (1936), Freedman (1980, chapter 5), Huang & Zhu (2005), Mukherji (2005) and Datta (2016).[3] According to the conventions of the literature on the predator-prey class of models, output per unit of effective labor, or the proxy for the level of economic activies might be interpreted as the prey whereas the proportion of infected persons might be interpreted as the predator. We should point out here that in specifying the general dynamical system in (4) we have not imposed any additional restrictions other than the ones required to describe our story of the emerging pandemic. In particular, we have not imposed any restrictions on the second and higher order derivatives of $f$ and $g$ so far.

## 2.2 Steady states and local stability

It is easy to see that the dynamical system represented by (4) has *four* steady states: ($a$) Trivial Equilibrium $E_1 : (0, 0)$, where both economic activities and infections collapse to zero; ($b$) 'Good Health Equilibrium' $E_2 : (\bar{y}_2, 0)$, where economic activities can continue with no infected persons; ($c$) Non-Trivial Equilibrium $E_3 : (\bar{y}_3, \bar{i}_3)$, where economic activities continue with a certain proportion of population remaining infected; and ($d$) 'Bad Equilibrium' $E_4 : (0, \bar{i}_4)$, where economic activities collapse even as a certain proportion of population remain infected. We

addition, cases of re-infection of COVID-19 have been reported, which makes emergence of herd immunity doubtful even in the long-run.

[3] In fact, we note that the specification of $g\left(0,0\right) > 0$ marks a departure of our model from the general form discussed in Mukherji (2005).

note that in general, multiple equilibria are possible only for the type $E_3$. Next, we propose the following:

**Proposition 1.** *$f(y,i) = 0$ is downward-sloping and intersects the y-axis and the i-axis at $\bar{y}_2$ and $\hat{i}$ respectively; $g(y,i)$ is downward-sloping and intersects the y-axis and the i-axis at $\hat{y}$ and $\bar{i}_4$ respectively; $\bar{y}_2, \hat{y}, \bar{i}_4, \hat{i} \in \Re_+$.*

*Proof.* $\left.\frac{di}{dy}\right|_{f(y,i)=0} = -\frac{f_y}{f_i} < 0$ and $\left.\frac{di}{dy}\right|_{g(y,i)=0} = -\frac{g_y}{g_i} < 0$ (given $f_y, f_i < 0$ and $g_y, g_i > 0$ from (2) and (3)), i.e. both $f(y,i) = 0$ and $g(y,i) = 0$ are downward-sloping. Further, given that $f(0,0) > 0$ and $f_y, f_i < 0$, from continuous differentiability of $f$ it follows that $\exists\ \bar{y}_2 \in \Re_+ \ni f(\bar{y}_2, 0) = 0$, and $\exists\ \hat{i} \in \Re_+ \ni f(0,\hat{i}) = 0$, i.e. $f(y,i) = 0$ intersects $y$ and $i$ axes in the non-negative quadrant. Similarly, given that $g(0,0) < 0$ and $g_y, g_i > 0$, from continuous differentiability of $g$ it follows that $\exists\ \hat{y} \in \Re_+ \ni g(\hat{y}, 0) = 0$ and $\exists\ \bar{i}_4 \in \Re_+ \ni g(0, \bar{i}_4) = 0$, i.e. $g(y,i) = 0$ intersects $y$ and $i$ axes in the non-negative quadrant. □

**Corollary 1.1.** *For the dynamical system represented by (4), $E_1$, $E_2$ and $E_4$ will always exist in the non-negative quadrant.*

*Proof.* Follows from proposition 1. □

**Proposition 2.** *$(\hat{y} - \bar{y}_2)(\hat{i} - \bar{i}_4) > 0$ is a sufficient condition for $E_3 \in \text{int } \Re^2_{++}$.*

*Proof.* $(\hat{y} - \bar{y}_2)(\hat{i} - \bar{i}_4) > 0 \iff$ either $\hat{y} > \bar{y}_2$ *and* $\hat{i} > \bar{i}_4$ or $\hat{y} < \bar{y}_2$ *and* $\hat{i} < \bar{i}_4 \iff f(y,i) = 0$ and $g(y,i) = 0$ must intersect each other at least once inside the non-negative quadrant. Hence it follows that there must be at least one interior non-trivial equilibrium represented by $E_3$. □

**Proposition 3.** *For the dynamical system represented by (4), the Trivial Equilibrium, $E_1$ is a saddle-point; the 'Good Health Equilibrium', $E_2$ is either locally stable or saddle-point; and the 'Bad Equilibrium', $E_4$ is either saddle-point or locally unstable.*

*Proof.* The jacobian of the dynamical system represented by (4) at the trivial equilibrium, $E_1$ is given by

$$J_1 = \begin{bmatrix} \alpha f\,(0,0) & 0 \\ 0 & \beta g\,(0,0) \end{bmatrix} \tag{5}$$

$\det(J_1) = \alpha\beta f\,(0,0)\,g\,(0,0) < 0$. Hence $E_1$ is always saddle-point. Similarly, at 'Good Health Equilibrium', $E_2$, the jacobian is given by

$$J_2 = \begin{bmatrix} \alpha f_y \bar{y}_2 & \alpha f_i \bar{y}_2 \\ 0 & \beta g\,(\bar{y}_2, 0) \end{bmatrix} \tag{6}$$

For $0 < \bar{y}_2 < \hat{y} \Leftrightarrow g\,(\bar{y}_2, 0) < 0$, we have $\det(J_2) = \alpha\beta f_y \bar{y}_2 g\,(\bar{y}_2, 0) > 0$ & $\text{trace}(J_2) = \alpha f_y \bar{y}_2 + \beta g\,(\bar{y}_2, 0) < 0$, i.e. $E_2$ is locally stable. For $\bar{y}_2 > \hat{y} > 0 \Leftrightarrow g\,(\bar{y}_2, 0) > 0$, we have $\det(J_2) < 0$, i.e. $E_2$ is a saddle-point. Finally, at the 'Bad Equilibrium', $E_4$, the jacobian is given by

$$J_4 = \begin{bmatrix} \alpha f\left(0, \bar{i}_4\right) & 0 \\ \beta g_y \bar{i}_4 & \beta g_i \bar{i}_4 \end{bmatrix} \tag{7}$$

For $0 < \bar{i}_4 < \hat{i} \Leftrightarrow f\left(0, \bar{i}_4\right) > 0$, we have $\det(J_4) = \alpha\beta g_i \bar{i}_4 f\left(0, \bar{i}_4\right) > 0$ & $\mathrm{trace}(J_4) = \alpha f\left(0, \bar{i}_4\right) + \beta g_i \bar{i}_4 > 0$, i.e. $E_4$ is locally stable. For $\bar{i}_4 > \hat{i} > 0 \Leftrightarrow f\left(0, \bar{i}_4\right) < 0$, we have $\det(J_4) < 0$, i.e. $E_4$ is a saddle-point. □

We might note from above discussion that the 'Good Health Equilibrium', $E_2$ might not always be attainable, in case the dynamics of the epidemiological process turns it into a saddle-point. In this case, the solution cannot converge to a steady state with no infection. However, without placing any additional restrictions on the higher order derivatives, we cannot make any further conclusion about the uniqueness or stability of the interior non-trivial equilibria. For this, we now turn to an example with specific functional forms.

### 2.3 Special Case: When $f$ and $g$ are linear in $y$ and $i$

We consider a special case of the generalized system we had earlier set up in (4), where $f$ and $g$ are both linear functions of $y$ and $i$ as follows:

$$\begin{aligned} \dot{y} &= \alpha\left(1 - \eta y - \kappa i\right) y \\ \dot{i} &= \beta\left(\theta y + \gamma i - 1\right) i \end{aligned} \tag{8}$$

We should point out here that $\kappa$ and $\theta$ are the two key variables which might be influenced through policy interventions. The policymaker might face a tradeoff between meeting the public health objective of controlling the pandemic and the economic objective of maintaining economic activities. $\kappa$ is an indicator of the relative emphasis placed by the policymaker on public health. Higher $\kappa$ might mean that the policymaker places a greater emphasis on public health objectives vis-à-vis the economic objectives, and is quick to respond to the spread of infections with restrictions on economic activities. $\theta$, on the other hand, indicates the effect of economic activities on transmission of the virus. This might largely depend on epidemiological factors (like how infectious the disease is); however, $\theta$ might be controlled to an extent through policy measures with large-scale use of safety measures including masks, sanitizers, provisions for social or physical distancing of workers in the workplaces etc. $\eta$ and $\gamma$, on the other hand, represent macroeconomic and epidemiological processes which, at least in the short-run within the time-period of progress of the pandemic, are largely beyond the control of policy interventions.

The dynamical system represented by (8) will have four steady states: $(a)$ Trivial equilibrium $E_1 : (0, 0)$; $(b)$ 'Good Health Equilibrium' $E_2 : \left(\frac{1}{\eta}, 0\right)$; $(c)$ Non-trivial

Equilibrium $E_3 : \left(\dfrac{\kappa - \gamma}{\kappa\theta - \gamma\eta}, \dfrac{\theta - \eta}{\kappa\theta - \gamma\eta}\right)$; and ($d$) 'Bad Equilibrium $E_4 : \left(0, \dfrac{1}{\gamma}\right)$. Note that with the imposition of linearity, now $f(y,i) = 0$ and $g(y,i) = 0$ can intersect each other at most once in the non-negative quadrant; in other words, there can be *at most* one instance of the non-trivial equilibrium, $E_3$, inside the non-negative quadrant. In line with proposition 1.1, $E_1$, $E_2$ and $E_4$ will always exist in the non-negative quadrant $Re_+ \times \Re_+$. In addition, the non-trivial equilibrium, $E_3$ will lie in the interior of positive quadrant $\Re_+ \times \Re_+$ provided

$$(\kappa - \gamma)(\theta - \eta) \geq 0 \tag{9}$$

There can be at most one such equilibrium. In other words, given the assumptions in the setup of (8) there will exist *at least three*, and *at most four* economically meaningful equilibria.

We represent, through phase diagrams, the dynamical system represented by (8). From the perspective of the local stability of the steady states, we can identify four distinct cases. These are shown below in figure 3.[4]

Next, we note the following about the stability properties of the steady states for the dynamical system represented by (8):

**Proposition 4.** *The Trivial Equilibrium, $E_1$ is always a saddle-point.*

*Proof.* The jacobian at $E_1$ is given by

$$A_1 = \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} \tag{10}$$

$\det(A_1) = -\alpha\beta < 0 \Rightarrow E_1$ is a saddle-point. □

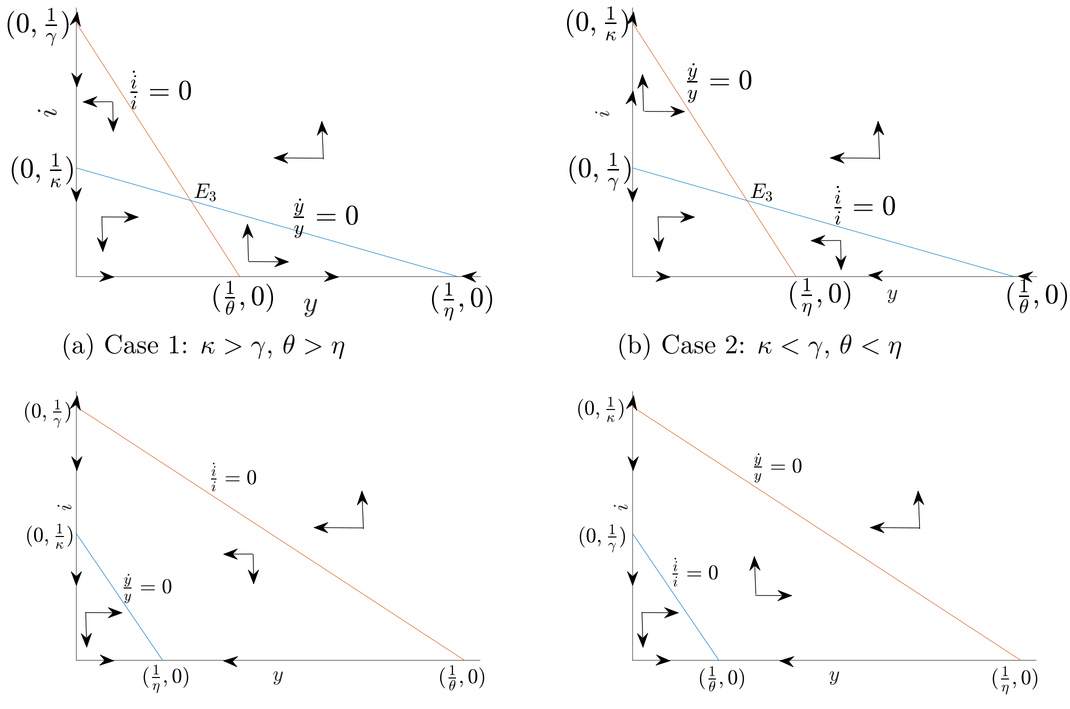
**Proposition 5.** *The 'Good Health Equilibrium', $E_2$ is saddle-point for $\theta > \eta$ (case 1 & 4 of figure 3) and locally stable for $\theta < \eta$ (case 2 & 3).*

*Proof.* The jacobian at $E_2$ is given by

$$A_2 = \begin{bmatrix} -\alpha & -\dfrac{\alpha\kappa}{\eta} \\ 0 & \beta\dfrac{\theta - \eta}{\eta} \end{bmatrix} \tag{11}$$

$\det(A_2) = -\alpha\beta\dfrac{\theta - \eta}{\eta} \gtreqless 0$ for $\theta \lesseqgtr \eta$, and $\text{trace}(A_2) = -\alpha + \beta\dfrac{\theta - \eta}{\eta} < 0$ for $\theta < \eta$. In other words, $E_2$ is saddle-point for $\theta > \eta$ and locally stable for $\theta < \eta$. □

[4] We should note here that figure 3c has been drawn with the assumption of $\kappa\theta < \gamma\eta$, i.e. $\dot{y}/y = 0$ is steeper than $\dot{i}/i = 0$. Alternate scenario where $\kappa\theta > \gamma\eta$ would make $\dot{y}/y = 0$ flatter than $\dot{i}/i = 0$, but will not change the stability properties of the steady states inside $\Re_+ \times \Re_+$ as long as $\gamma < \kappa$ and $\theta < \eta$, specifying the relative positions of the $y$ and the $i$ intercepts. Same argument might be made for case 3d, drawn with the assumption of $\kappa\theta > \gamma\eta$, making $\dot{y}/y = 0$ flatter than $\dot{i}/i = 0$. Alternate assumptions about the relative slopes of the curves will not affect the stability properties of the steady states inside the non-negative quadrant as long as the relative intercepts remain the same.




(a) Case 1: $\kappa > \gamma$, $\theta > \eta$

(b) Case 2: $\kappa < \gamma$, $\theta < \eta$

(c) Case 3: $\kappa > \gamma$, $\theta < \eta$

(d) Case 4: $\kappa < \gamma$, $\theta > \eta$

Figure 3: Four cases

As we noted earlier, $\theta$ depends on a combination of epidemiological factors and policy interventions. The policymaker interested in attaining the 'Good Health Equilibrium' with no infections, might attempt to reduce $\theta$ through facilitating safety practices like use of masks, sanitizers, physical distancing of workers at workplaces etc. However, in case of a particularly aggressive epidemic due to the infectious nature of the virus itself, it might not be possible to reduce $\theta$ beyond a certain level. In this case, it will not be possible for the policymaker to pursue policies to attain the 'Good Health Equilibrium'.

Let us now turn our attention to the non-trivial equilibrium, $E_3$. We recall from (9) that $E_3 \in \text{int } \Re^2_{++}$ under two conditions: case 1 (figure 3a) where $\kappa > \gamma$ & $\theta > \eta$ and case 2 (figure 3b) where $\kappa < \gamma$ & $\theta < \eta$. Next, we note the following:

**Proposition 6.** *In case 2 (figure 3a) where $\kappa < \gamma$ & $\theta < \eta$, the non-trivial equilibrium $E_3 \in \text{int } \Re^2_{++}$, is saddle-point.*

*Proof.* The jacobian at $E_3$ is given by

$$A_3 = \begin{bmatrix} -\alpha\eta\dfrac{\kappa-\gamma}{\kappa\theta-\gamma\eta} & -\alpha\kappa\dfrac{\kappa-\gamma}{\kappa\theta-\gamma\eta} \\ \beta\theta\dfrac{\theta-\eta}{\kappa\theta-\gamma\eta} & \beta\gamma\dfrac{\theta-\eta}{\kappa\theta-\gamma\eta} \end{bmatrix} \tag{12}$$

$\det(A_3) = \alpha\beta\left(\kappa\theta - \gamma\eta\right)\left(\frac{\kappa-\gamma}{\kappa\theta-\gamma\eta}\right)\left(\frac{\theta-\eta}{\kappa\theta-\gamma\eta}\right)$ and $\mathrm{trace}(A_3) = -\alpha\eta\frac{\kappa-\gamma}{\kappa\theta-\gamma\eta} + \beta\gamma\frac{\theta-\eta}{\kappa\theta-\gamma\eta}$. In case 2, it is clear that $\det(A_3) < 0$, i.e. $E_3$ is saddle-point. □

**Corollary 6.1.** *In case 1 (figure 3a) where $\kappa > \gamma$ & $\theta > \eta$, the non-trivial equilibrium $E_3 \in \mathrm{int}\ \Re^2_{++}$ undergoes a non-degenerate Andronov-Hopf bifurcation for a critical value of the speed of adjustment parameter, $\beta$ at $\hat{\beta}$. $E_3$ is locally stable $\forall\ \beta < \hat{\beta}$.*

*Proof.* In case 1, it is clear that $\kappa\theta - \gamma\eta > 0 \Leftrightarrow \det(A_3) > 0$. Further, defining

$$\hat{\beta} = \frac{\alpha\eta\left(\kappa-\gamma\right)}{\gamma\left(\theta-\eta\right)} \tag{13}$$

we have $\mathrm{trace}(A_3) < 0\ \forall\ \beta < \hat{\beta}$. Further, while perturbing $\beta$ through $\hat{\beta}$, we find that $\mathrm{trace}(A_3) = 0$ at $\beta = \hat{\beta}$ and $\frac{\partial \mathrm{trace}(A_3)}{\partial\beta} = \gamma\left(\theta-\eta\right) > 0$, i.e. the derivative exists, is smooth and differentiable. This satisfies both the existence and the transversality condition for the existence of a non-degenerate Andronov-Hopf bifurcation. □

**Proposition 7.** *The 'Bad Equilibrium', $E_4$ is either saddle-point or locally unstable.*

*Proof.* The jacobian at $E_4$ is given by

$$A_4 = \begin{bmatrix} \frac{\alpha\left(\gamma-\kappa\right)}{\gamma} & 0 \\ \frac{\beta\theta}{\gamma} & \beta \end{bmatrix} \tag{14}$$

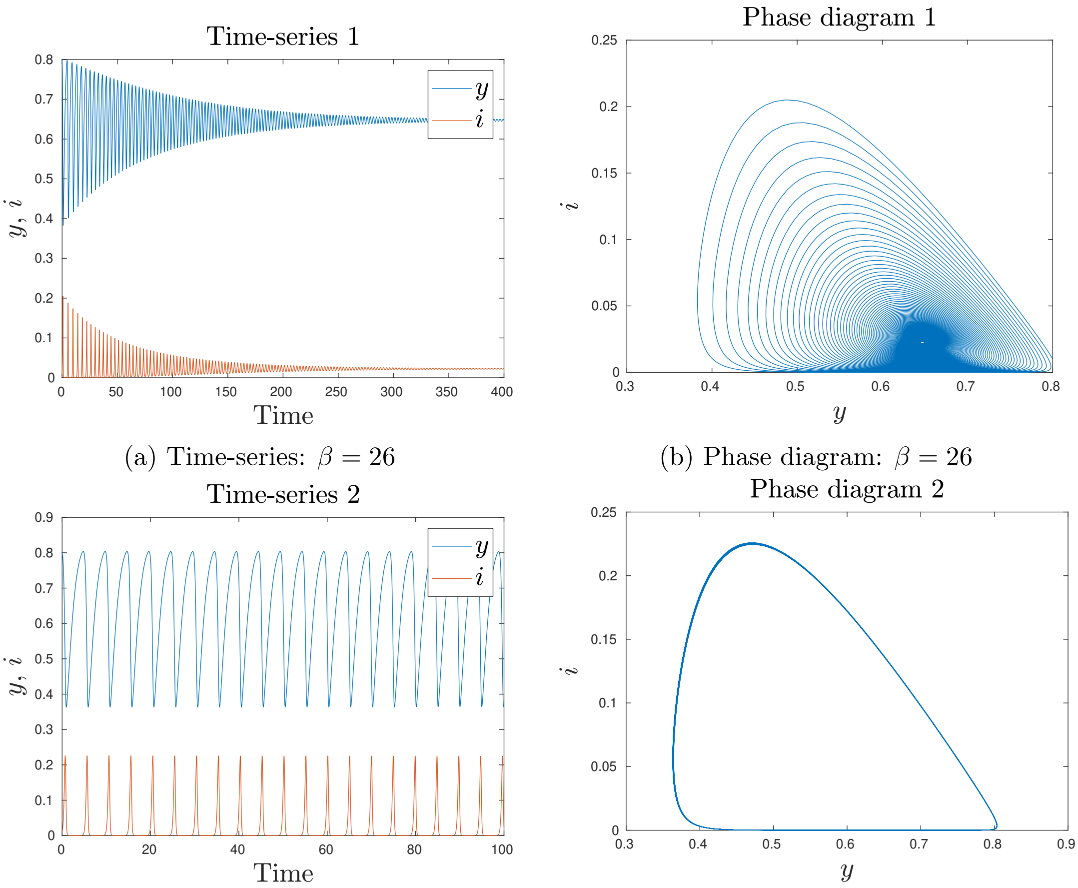
$\det(A_4) = \alpha\beta\frac{\gamma-\kappa}{\gamma}$ and $\mathrm{trace}(A_4) = \frac{\alpha\left(\gamma-\kappa\right)}{\gamma} + \beta$. It follows that for $\gamma < \kappa \Leftrightarrow \det(A_4) < 0$, while for $\gamma > \kappa \Leftrightarrow \mathrm{trace}(A_4) > 0$, i.e. $E_4$ is either saddle-point or locally unstable. □

To summarize, the dynamical system represented by (8) has four steady states, out of which the Trivial Equilibrium, $E_1$, is always a saddle-point; the 'Good Health Equilibrium', $E_2$, is saddle-point in case 1 and 4, and locally stable in case 2 and 3; the non-trivial equilibrium, $E_3$ is saddle-point in case 2 and might be stable or unstable in case 1 where it undergoes an Andronov-Hopf bifurcation; the 'Bad Equilibrium' is either saddle-point or locally unstable.

We demonstrate these possibilities with a numerical example.[5] Consider the following parameter configuration: $\gamma = 1.3$; $\theta = 1.5$; $\kappa = 10$; $\eta = 1.2$; $\alpha = 1$. The steady states will then be at $E_1 : (0, 0)$, $E_2 : (0.8333, 0)$, $E_3 : (0.6473, 0.0223)$ and $E_4 : (0, 0.7692)$. Starting from an initial point of $(0.8, 0.1)$, we find that the solution converges to the non-trivial equilibrium for low values of $\beta$, but at $\beta = 26.7692$ we

[5]All numerical analyses in this study are performed using Matlab version R2018a (9.4.0.813654).

find that $E_3$ undergoes a non-degenerate Andronov-Hopf bifurcation leading to emergence of limit cycles. This is shown using time-series and phase diagrams in figure 4 below.




(a) Time-series: $\beta = 26$

(b) Phase diagram: $\beta = 26$

(c) Time-series: $\beta = 26.7692$

(d) Phase diagram: $\beta = 26.7692$

Figure 4: Time-series and phase diagram of convergence & limit cycles

Cyclical possibilities from Andronov-Hopf bifurcation are indicative of waves of infection and economic downturns, similar to the empirical stylized facts we saw in figure 2. Consider a situation where the pandemic arrives in a country resulting in growth of infections. Let the proportion of population which is infected, $i$ grow at a speed (represented by $\beta$) such that the non-trivial equilibrium undergoes Andronov-Hopf bifurcation. The spread of infections force the policymakers to put in place restrictions on economic activities like travel restrictions, lockdowns etc. resulting in a reduction in the growth of economic activities, $y$. A reduction in the growth of $y$, from the equation of motion for $i$, eventually controls the growth rate of $i$, leading to a temporary halt in the spread of the pandemic. However, the reduction in the growth of economic activities, along with a temporary halt in the spread of the pandemic, induces the policymaker to lift some of the restrictions placed earlier, in order to boost economic activities once again. This eventually leads to a turnaround in the economic downturn, which in turn, once again creates conditions for the beginning of another wave of infections. Logically, this mechanism is similar to the classic predator-prey type of dynamics described by the Kolmogorov-Lotka-Volterra framework, which might result in repeated waves of infection interspersed with phases of economic downturns.

# 3 Comparative Dynamics and Policy Implications

In this section, we explore the dynamic effect of changes in the parameters on the solutions. In particular, we consider the effect of changes in $\kappa$ and $\theta$, which to an extent, might be influenced by policy.

We note, first of all, that the attainment of the 'Good Health Equilibrium' ($E_2$) might not always be possible. We noted earlier that $E_2$ is locally stable only if $\theta < \eta$. Policy interventions like use of safety measures at workplace and other public places might reduce $\theta$ to some extent, but in case of a highly infectious disease like COVID-19, this might not be adequate to reduce $\theta$ sufficiently so as to satisfy the stability condition. If this is the case, then the policy interventions must look beyond attempts to attain $E_2$.

If $\theta > \eta$, so that attainment of $E_2$ is not dynamically feasible, then the policy-maker will have to turn the attention to $E_3$, the next best scenario. Note that in this case, one is either in case 1 (if $\kappa > \gamma$) or case 4 (if $\kappa < \gamma$) of figure 3. The infections explode in case 4, so to avoid explosion in infections $\kappa$ need to be kept higher than $\gamma$. In other words, the non-pharmaceutical policy interventions to control the pandemic must defeat the natural growth parameter of the pandemic represented by $\gamma$.

Even if $\kappa > \gamma$ to ensure that the non-trivial equilibrium, $E_3$ exists in the positive quadrant, this might not be enough. We need to further ensure that it is stable so that the solutions converge to it. This requires that the speed of adjustment of infection growth rate, $\beta$ is lower than $\hat{\beta}$ given by (13). Higher the $\hat{\beta}$, higher are the possibilities of the solutions converging to the non-trivial equilibrium $E_3$ (i.e. avoiding explosive growth in infections). We note that

$$\frac{\partial \hat{\beta}}{\partial \kappa} = \frac{\alpha\eta}{\gamma(\theta - \eta)} > 0 \tag{15}$$

$$\frac{\partial \hat{\beta}}{\partial \theta} = -\frac{\alpha\eta(\kappa - \gamma)}{\gamma(\theta - \eta)^2} < 0 \tag{16}$$

In other words, convergence of infections as well as economic activities to $E_3$ requires a combination of policy interventions to increase $\kappa$ as well as to reduce $\theta$. The former would require policy restrictions being placed on economic activities, whereas the latter would require putting in place safety measures so that the economic activities, albeit at reduced level, can be performed safely while keeping the spread of infection at check.

In case $E_3$ is locally stable (i.e. can be attained), we can perform comparative static exercise on $E_3$ to find that

$$\frac{\partial \bar{y}_3}{\partial \kappa} = \frac{\gamma(\theta - \eta)}{(\kappa\theta - \gamma\eta)^2} > 0 \tag{17}$$

$$\frac{\partial \bar{y}_3}{\partial \theta} = -\frac{\kappa(\kappa - \gamma)}{(\kappa\theta - \gamma\eta)^2} < 0 \tag{18}$$

$$\frac{\partial \bar{i}_3}{\partial \kappa} = -\frac{\theta(\theta - \eta)}{(\kappa\theta - \gamma\eta)^2} < 0 \tag{19}$$

$$\frac{\partial \bar{i}_3}{\partial \theta} = \frac{\eta(\kappa - \gamma)}{(\kappa\theta - \gamma\eta)^2} > 0 \tag{20}$$

In other words, a combination of an increase in $\kappa$ and a reduction in $\theta$ can achieve, in addition to stability of $E_3$, an increase in $y$ and a reduction in $i$ in the steady state. In other words, non-pharmaceutical policy interventions can increase economic activities and reduce the proportion of persons infected in the steady state.

# 4 Conclusions

We can summarize the discussion in the previous sections as follows: at the onset of the pandemic, the first policy response might be to fully control the pandemic and attain the 'Good Health Equilibrium'. This might require policy interventions to keep $\theta$ low, i.e. undertake safety measures while performing economic activities to prevent the spread of infection. In case of a highly infectious disease like COVID-19, this might not be adequate to keep $\theta$ low. If this is the case (i.e. if $E_2$ is not stable), then there need to be additional policy interventions, in the form of restrictions on economic activities, to increase $\kappa$ in order to ensure that ($a$) the non-trivial equilibrium, $E_3$ exists in the positive quadrant, and ($b$) is stable. The first would require keeping $\kappa$ high enough to exceed $\gamma$, i.e. impose restrictions on economic activities which are strong enough to keep up with the natural rate of growth of infection, while the second would require a combination of high $\kappa$ and low $\theta$. In short, preventing an explosion in infection as well as a collapse in economic activities would require a combination of two types of non-pharmaceutical policy interventions, firstly, safety measures at workplace and other public places so that economic activities can be conducted safely without spreading infection; and secondly, some form of restrictions on economic activities. Typically, any one of these in isolation will not be enough to prevent either an explosion in infection or a collapse in economic activities.

A failure to control the progress of the pandemic in the manner outlined above might lead to two kinds of adverse dynamic outcomes: it might either result in explosive dynamic outcomes leading to uncontrolled spread of the pandemic (resulting in massive loss of human lives) or there might be cyclical trajectories from Andronov-Hopf bifurcation resulting in repeated waves of infection and very short-run cycles of fluctuations in economic activities. The stylized facts shown in figure 2 of section 1 might be a possible outcome of such dynamic processes. Continuation of such cyclical trajectories would result in substantial welfare losses, both in terms of loss of human lives to infection as well as loss of livelihood from widespread economic displacement due to such short-run fluctuations in economic activities.

Finally, we should be careful in mechanically interpreting the policy recommendations of our model in terms of increase in $\kappa$ and reducing $\theta$. We have already discussed above the limits of reducing $\theta$ beyond a certain level, given that $\theta$ is to some extent determined by the epidemiological dynamics of the specific disease beyond the control of the policymaker. In case of a highly infectious disease like COVID-19, it might not be possible to reduce $\theta$ beyond a certain point at least in the short-run. At the same time, there might be political limits on increasing $\kappa$ as it involves restrictions on economic activities. It might not be politically feasible for any government or the policymaker to increase $\kappa$ beyond a point as the immediate impact of this in the form of loss in livelihood of workers, as well as damage to the business interests might result in political pressure on the government to reverse these policies. In addition, we should also note that we have no particular reason to assume that either of the steady states, $E_2$ and $E_3$ represent a socially or politically desirable level of economic activities. If the steady state level of economic activies, $y$ fall short of its socio-politically desired level (for instance, in case it is associated with a larger than desired rate of unemployment) then the policymakers might be tempted to expand economic activities, which might act as a further disincentive to impose restrictions during pandemic, destabilizing the resulting dynamics further.

In the long-run, therefore, prevention of the pandemic might require advanced planning prior to the onset of the pandemic, putting in place public health infrastructure as well as institutional mechanism to protect livelihoods at times when economic activities need to be curtailed to prevent the spread of pandemic. In a globalized world, where international travel contributes to spread of the pandemic, this might also require international cooperation among countries to jointly put in place an institutional infrastructure to enable this. The simplified model presented here is not in a position to incorporate many of these factors. We plan to take up some of these issues in future extensions of this work. The simple model presented here, therefore, might be looked upon as a preliminary enquiry into the dynamics of interaction between the epidemiological processes underlying the COVID-19 pandemic and the macroeconomic processes which affect the policy interventions to control the pandemic.

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